Convexity and Concavity

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1 Introduction

In this article we are going to learn basic inequality of convex and linear function .These methods are useful in finding the maxima of some convex functions. In fact a powerful technique which is an extension of this method of convexity can kill some tough irregular inequalities with ease. These techniques are not very well known inspite of that it can solve inequalities from IMO and USAMO level easily.In this article I will present a proof of Schur's inequality for $r = 1$.

DEFINITION

(It's quite boring :P, but important too) :-

* A real - valued function defined on the interval is linear if the graph of the function is a straight line.

* A real - valued function defined on the interval is convex if line segment formed by joining any two points on the graph of the function lies above or on the graph. In fact the graph looks like a convex lens.

* All linear functions are convex(as well as concave) but the converse doesn't hold

Counter example $f(x) = x^2$

* A function f(x) is a convex function if the second derivative of the function is non -negative. In other words $f''(x) \geq 0$.

Now we give 2 basic inequalities on linear and convex functions

1. Any linear function attains both of it's extrema at the end point of the interval.

2. Any convex function attains it's maxima at the endpoint of the interval.

Though the above two inequalities looks trivial and easy but are able to kill hard inequalities instantly if used wisely.

PROBLEMS

We start with 2 trivial problems to give an idea on how to use this method!!

Problem 1.

Let a, b, c be non negative real numbers satisfying $a + b + c = 1$ .Find the maximum value of $a^2 + b^2 + c^2$

Solution.

"If we try AM - GM , Cauchy Schwarzs directly we obtain minima of $a^2 + b^2 + c^2$,But what about the maxima , here is where we employ our method !!."

First of all See that a, b, c is defined in the interval $[0, 1]$

See that $a^2 + b^2 + c^2 = (1 - b - c)^2 + b^2 + c^2 = 2b^2 + 2c^2 - 2b - 2c + 2bc + 1$

Consider the above expression as a function of b i.e.

$f(b) = 2b^2 + b(2c - 2) + 2c^2 - 2c + 1$, see that $f''(b) = 4 > 0$

so , The above expression is a convex function for b and similarly for c.

Hence the maxima of $2b^2 + 2c^2 - 2b - 2c + 2bc + 1$ occurs when b, c

are at the endpoints Of the interval i.e $(b, c) = (0, 1), (0, 0), (1, 0)$

If $b, c = (0, 0)$ or $(0, 1)$ or $(1, 0)$ then $2b^2 + 2c^2 - 2b - 2c + 2bc + 1 = 1$

So, $a^2 + b^2 + c^2 \leq 1$. With equality at $(a, b, c) = (0, 0, 1)$ or any of it's permutation.

Problem 2.

Let a, b, c, d lie in the interval $[0, 1]$. Prove that $(1 - a)(1 - b)(1 - c)(1 - d) + a + b + c + d \geq 1$.

Solution.

If we see the above expression as a function in "a", then we realize that it is of form $k(1-a)+a+m$, where k, m are constants , hence it is a linear function.

So, it's minima (as well as maxima) occurs when $(a, b, c, d) = 0$ or 1 , we can now check the value of the expression at these points and conclude the result.

Now we move to a tough problem from USAMO - 1980.(This problem shows the power of this method)

Problem 3.

Prove that if a, b, c lie on the interval $[0, 1]$, then $\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1$

Solution

let $f(a, b, c) = \frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c)$

If we differentiate twice $f(a, b, c)$ w.r.t "a",

$=> \frac{d^2 f(a,b,c)}{da^2} = \frac{2b}{(c+a+1)^3} + \frac{2c}{(a+b+1)^3} \geq 0$

So $f(a, b, c)$ is a convex function.

Hence it's maxima occurs when (a,b,c) = 0 or 1 ,we can now check the value of $f(a, b, c)$ at these 8 points.

If we use AM - GM,Cauchy Schwarz,or any standard technique then it would have required a considerable amount of manipulation and computation while the above solution was short and easy.

Now, most of you must be familiar with the Schur's inequality (especially when $r = 1$)

We will provide a proof of Schur's for $r = 1$ using this method.

Problem 4.

Let a, b, c be three non-negative real numbers , prove that $a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(b + c) + ca(c + a)$

(This is Schur's inequality for $r = 1$!)

Solution.

See that the above inequality is homogeneous. So, We may assume $a+b+c = 1$

By manipulating the desired inequality, we need to prove

$9abc + 1 \geq 4(ab + bc + ca)$

Here we have used

$a^3 + b^3 + c^3 - 3abc = (a + b + c)^3 - 3(ab + bc + ca)(a + b + c)$

Now, See that $a = 1 - b - c$
So, it remains to prove that

$9(1 - b - c)bc + 1 \geq 4((1 - b - c)(b + c) + bc)$

$=> bc(5 - 9(b + c)) + 4(b + c)^2 - 4(b + c) + 1 \geq 0$

Let $w = bc$ and let $f(w) = w(5 - 9(b + c)) + 4(b + c)^2 - 4(b + c) + 1$

See that we need to prove $w(5 - 9(b + c)) + 4(b + c)^2 - 4(b + c) + 1 = f(w) \geq 0$

As $f(w)$, is a linear function in "w", So it's minima occurs at the endpoints of the interval!!

But now how do we determine the endpoints of $w = bc$?

See that firstly $bc \geq 0$ also AM-GM gives , $bc \leq \frac{(b+c)^2}{4}$

Hence we have found the boundary points !!

For the sake of convenience Let $x = b + c$
Therefore,

$f(w) = w(5 - 9x) + 4x^2 - 4x + 1$

Also,

$f(0) = 4x^2 - 4x + 1 = (2x - 1)^2 \geq 0$

$f(\frac{x^2}{4}) = \frac{-9x^3+21x^2-16x+4}{4} = \frac{(1-x)(9x^2-12x+4)}{4}$

$= \frac{(1-x)(3x-2)^2}{4} \geq 0$, because $1 - x = a \geq 0$

As, we checked Both the endpoint , we are done.

The above solution was different from other's .Instead of considering a function in b or c ,We fixed the sum $b + c$ and bounded bc. The above is a powerful

expansion of our method .

In fact it created a linear function which made it easier to deal with the Minima.

Now we solve a problem from IMO – 1984 ,

Problem 5.

Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where $x, y \text{and} z$ are non-negative real numbers satisfying $x + y + z = 1$.

Solution 5.

See that $0 \leq yz + zx + xy - 2xyz = yz + (1 - y - z)(y + z) - 2(1 - y - z)(yz) = (2(y + z) - 1)yz + (1 - y - z)(y + z)$

Let $y + z = a$, $yz = b$ Also define $f(b) = (2a - 1)b + a - a^2$,

We need to prove that $0 \leq f(b) \leq \frac{7}{27}$,

Also see that $f(b)$ is a linear function. So it's minima as well as maxima occurs at the endpoint Of the intervals.

Now see that $0 \leq b \leq \frac{a^2}{4}$.

Also,

$f(0) = 0 \leq a - a^2 \leq \frac{1}{4}$, because

(i)a–$a^2 - \frac{1}{4} = -(a - \frac{1}{2})^2$

(ii)$a - a^2 = a(1 - a) \geq 0$ as $a = 1 - x \geq 0$

Now,

$f(\frac{a^2}{4}) = (2a - 1)(\frac{a^2}{4}) + a$–$a^2 = \frac{2a^3 - 5a^2 + 4a}{4}$,

See that $\frac{2a^3 - 5a^2 + 4a}{4} = \frac{(a)(2(a - \frac{5}{4})^2 + \frac{7}{8})}{4} > 0$

And ,

$\frac{7}{27} - \frac{2a^3 - 5a^2 + 4a}{4} = \frac{(3a - 2)^2(7 - 6a)}{108} \geq 0$,

Because $a < \frac{7}{6}$.

As we have checked both the endpoints , we are done

Now These are some suggestions while using this method :-

1. Always check that whether the function is a convex or not before applying this method .

2. This method is helpful in finding the maxima of convex function.

3. If you see that the equality case of an inequality are at the endpoints of the interval , then you should try this method .

4. Fixing "$b + c$" and bounding "bc" to form a linear equation can prove to be helpful .

5. If you directly attack an inequality By Langrage Multipliers, then you have to go through a system of equations.While this method helps in skipping that part in case of a convex or a linear function

EXERCISES

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1.Let $a_1, a_2, ..., a_{15}$ be real numbers from the interval $[-x, x]$ where x is a positive real number. Determine the minimum value of $a_1a_2 + a_2a_3 + ... + a_{15}a_1$ in terms of x.

2.Prove that $x^2 + y^2 + z^2 \leq xyz + 2$ where x, y, z is defined in the interval $[0, 1]$.

3.a, b, c are non-negative reals such that $a + b + c = 1$. Prove that $a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4}$

[Hint , eliminate "a" in the above expression and form a function in bc]

4.Prove Kantorovich's Inequality (You can search for it on wikipedia)

5.Prove that the area of a triangle lying inside the unit square does not exceed $\frac{1}{2}$

REFERENCES

:-

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2.T. Andreescu and R. Gelca, Mathematical Olympiad Challenges

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